

Some Generalizations of the Eneström-Kakeya Theorem

Eze R. Nwaeze

Department of Mathematics and Statistics

Auburn University

Auburn, AL 36849, USA

e-mail: ern0002@auburn.edu

Abstract

Let $p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n$ be a polynomial of degree n , where the coefficients a_j , $j \in \{0, 1, 2, \dots, n\}$, are real numbers. We impose some restriction on the coefficients and then prove some extensions and generalizations of the Eneström-Kakeya Theorem.

Keywords: Real polynomials; Location of zeros; MATLAB.

2010 Mathematics Subject Classification: 30C10, 30C15

1 Introduction

A classical result due to Eneström [5] and Kakeya [7] concerning the bounds for the moduli of zeros of polynomials having positive coefficients is often stated as:

Theorem A. Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial with real coefficients satisfying

$$0 < a_0 \leq a_1 \leq a_2 \leq a_3 \dots \leq a_n.$$

Then all the zeros of $p(z)$ lie in $|z| \leq 1$.

In the literature there exist several extensions and generalizations of this result (see [1], [2], [6] and [8]). Joyal et al. [6] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily nonnegative. In fact, they proved the following result:

Theorem B. Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial degree n , with real coefficients satisfying

$$a_0 \leq a_1 \leq a_2 \leq a_3 \dots \leq a_n.$$

Then all the zeros of $p(z)$ lie in the disk

$$|z| \leq \frac{1}{|a_n|}(a_n - a_0 + |a_0|).$$

Aziz and Zargar [4] relaxed the hypothesis in several ways and among other things proved the following result:

Theorem C. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,*

$$0 < a_0 \leq a_1 \leq a_2 \leq a_3 \dots \leq ka_n.$$

Then all the zeros of $p(z)$ lie in the disk

$$|z + k - 1| \leq k.$$

In 2012, they further generalized Theorem C which is an interesting extension of Theorem A. In particular, they [3] proved the following results:

Theorem D. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . If for some positive numbers k and ρ with $k \geq 1$, $0 < \rho \leq 1$,*

$$0 \leq \rho a_0 \leq a_1 \leq a_2 \leq a_3 \dots \leq ka_n,$$

then all the zeros of $p(z)$ lie in the disk

$$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho).$$

Theorem E. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . If for some positive number ρ , $0 < \rho \leq 1$, and for some nonnegative integer λ , $0 \leq \lambda < n$,*

$$\rho a_0 \leq a_1 \leq a_2 \leq a_3 \dots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \dots \geq a_{n-1} \geq a_n,$$

then all the zeros of $p(z)$ lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \left[2a_\lambda - a_{n-1} + (2 - \rho)|a_0| - \rho a_0 \right].$$

Looking at Theorem D, one might want to know what happens if ρa_0 is NOT nonnegative. In this paper we prove some extensions and generalization of Theorems D and E which in turns gives an answer to our enquiry.

2 Main Results

Theorem 1. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . If for some real numbers α and β ,*

$$a_0 - \beta \leq a_1 \leq a_2 \leq \dots \leq a_n + \alpha,$$

then all the zeros of $p(z)$ lie in the disk

$$\left| z + \frac{\alpha}{a_n} \right| \leq \frac{1}{|a_n|} \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right].$$

If $\alpha = (k-1)a_n$ with $k \geq 1$ and $\beta = (1-\rho)a_0$ with $0 < \rho \leq 1$, we get the following.

Corollary 1. Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . If for some positive numbers $k \geq 1$ and ρ , with $0 < \rho \leq 1$,

$$\rho a_0 \leq a_1 \leq a_2 \leq \dots \leq k a_n,$$

then all the zeros of $p(z)$ lie in the disk

$$|z + k - 1| \leq \frac{1}{|a_n|} \left[(k a_n - \rho a_0) + |a_0|(2 - \rho) \right].$$

If $a_0 > 0$, then Corollary 1 amounts to Theorem D.

Theorem 2. Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . If for some real number s and for some positive integer λ , $0 < \lambda < n$

$$a_0 - s \leq a_1 \leq a_2 \leq a_3 \dots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \dots \geq a_{n-1} \geq a_n,$$

then all the zeros of $p(z)$ lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \left[2a_\lambda - a_{n-1} + s - a_0 + |s| + |a_0| \right].$$

.

If we take $s = (1-\rho)a_0$, with $0 < \rho \leq 1$, then Theorem 2 becomes Theorem E. Instead of proving Theorem 2, we shall prove a more general case. In fact, we prove the following result:

Theorem 3. Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . If for some real number t , s and for some positive integer λ , $0 < \lambda < n$

$$a_0 - s \leq a_1 \leq a_2 \leq a_3 \dots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \dots \geq a_{n-1} \geq a_n + t,$$

then all the zeros of $p(z)$ lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} - \left(1 + \frac{t}{a_n} \right) \right| \leq \frac{1}{|a_n|} \left[2a_\lambda - a_{n-1} + s - a_0 + |s| + |a_0| + |t| \right].$$

.

3 Proof of the Theorems

Proof of Theorem 1. Consider the polynomial

$$g(z) = (1 - z)p(z)$$

$$\begin{aligned} &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} - \alpha z^n + (a_n + \alpha - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + \beta)z - \beta z + a_0 \\ &= -z^n(a_n z + \alpha) + (a_n + \alpha - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + \beta)z - \beta z + a_0 \\ &= -z^n(a_n z + \alpha) + \phi(z), \end{aligned}$$

where

$$\phi(z) = (a_n + \alpha - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + \beta)z - \beta z + a_0.$$

Now for $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| &\leq |a_n + \alpha - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_1 - a_0 + \beta| + |\beta| + |a_0| \\ &= a_n + \alpha - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_1 - a_0 + \beta + |\beta| + |a_0| \\ &= a_n + \alpha - a_0 + \beta + |\beta| + |a_0|. \end{aligned}$$

Since this is true for all complex numbers with a unit modulus, then it must also be true for $1/z$. With this in mind, we have

$$|z^n \phi(1/z)| \leq a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \quad \forall z : |z| = 1. \quad (1)$$

Also, the function $\Phi(z) = z^n \phi(1/z)$ is analytic in $|z| \leq 1$, hence, Inequality (1) holds inside the unit circle by the Maximum Modulus Theorem. That is,

$$|\phi(1/z)| \leq \frac{a_n + \alpha - a_0 + \beta + |\beta| + |a_0|}{|z|^n} \quad \forall z : |z| \leq 1.$$

Replacing z by $1/z$, we get

$$|\phi(z)| \leq \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right] |z|^n \quad \forall z : |z| \geq 1.$$

Now for $|z| \geq 1$, we obtain

$$\begin{aligned}
|g(z)| &= |-z^n(a_n z + \alpha) + \phi(z)| \\
&\geq |z^n||a_n z + \alpha| - |\phi(z)| \\
&\geq |z^n||a_n z + \alpha| - \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right] |z|^n \\
&= |z^n| \left(|a_n z + \alpha| - \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right] \right) \\
&> 0
\end{aligned}$$

if and only if

$$|a_n z + \alpha| > \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right]$$

if and only if

$$\left| z + \frac{\alpha}{a_n} \right| > \frac{1}{|a_n|} \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right].$$

Thus, all the zeros of $g(z)$ whose modulus is greater than or equal to 1 lie in

$$\left| z + \frac{\alpha}{a_n} \right| \leq \frac{1}{|a_n|} \left[a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \right]. \quad (2)$$

But those zeros of $p(z)$ whose modulus is less than 1 already satisfy (2) - since $|\phi(z)| \leq a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \quad \forall z : |z| = 1$ and $\phi(z) = g(z) + z^n(a_n z + \alpha)$. Also, all the zeros of $p(z)$ are zeros of $g(z)$. That completes the proof of Theorem 1. \square

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned}
g(z) &= (1 - z)p(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} \\
&\quad + (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0)z + a_0 \\
&= -z^n[a_n z - a_n + a_{n-1} - t] - tz^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} \\
&\quad + (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0 + s)z - sz + a_0 \\
&= -z^n[a_n z - a_n + a_{n-1} - t] + \psi(z),
\end{aligned}$$

where

$$\psi(z) = -tz^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0 + s)z - sz + a_0.$$

For $|z| = 1$, we get

$$\begin{aligned} |\psi(z)| &\leq |t| + |a_{n-1} - a_{n-2}| + \cdots + |a_{\lambda+1} - a_\lambda| + |a_\lambda - a_{\lambda-1}| + \cdots + |a_1 - a_0 + s| + |s| + |a_0| \\ &= |t| + a_{n-2} - a_{n-1} + \cdots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + \cdots + a_1 - a_0 + s + |s| + |a_0| \\ &= |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|. \end{aligned}$$

It is clear that

$$|z^n \psi(1/z)| \leq |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \quad (3)$$

on the unit circle. Since the function $\Psi(z) = z^n \psi(1/z)$ is analytic in $|z| \leq 1$, Inequality (3) holds inside the unit circle by the Maximum Modulus Theorem. That is,

$$|\psi(1/z)| \leq \frac{|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|}{|z|^n}$$

for $|z| \leq 1$. Replacing z by $1/z$ we get

$$|\psi(z)| \leq \left[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \right] |z|^n$$

for $|z| \geq 1$.

Now for $|z| \geq 1$, we have

$$\begin{aligned} |g(z)| &\geq |z^n| |a_n z - a_n + a_{n-1} - t| - |\psi(z)| \\ &\geq |z^n| |a_n z - a_n + a_{n-1} - t| - \left[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \right] |z|^n \\ &= |z^n| \left(|a_n z - a_n + a_{n-1} - t| - \left[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \right] \right) \\ &> 0 \end{aligned}$$

if and only if

$$|a_n z - a_n + a_{n-1} - t| > \left[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \right]$$

if and only if

$$\left| z + \frac{a_{n-1}}{a_n} - \left(1 + \frac{t}{a_n} \right) \right| > \frac{1}{|a_n|} \left[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \right].$$

Hence, the zeros of $p(z)$ with modulus greater or equal to 1 are in the closed disk

$$\left| z + \frac{a_{n-1}}{a_n} - \left(1 + \frac{t}{a_n} \right) \right| \leq \frac{1}{|a_n|} \left[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \right].$$

Also, those zeros of $p(z)$ whose modulus is less than 1 already satisfy the above inequality since $\psi(z) = g(z) + z^n[a_n z - a_n + a_{n-1} - t]$, and for $|z| = 1$, $|\psi(z)| \leq |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|$. That completes the proof. \square

4 Demonstrating Examples

Example 1. Let's consider the polynomial

$$p(z) = 3z^5 + 4z^4 + 3z^3 + 2z^2 + z - 1.$$

The coefficients here are $a_5 = 3$, $a_4 = 4$, $a_3 = 3$, $a_2 = 2$, $a_1 = 1$ and $a_0 = -1$. We cannot apply Theorems A, B, C and D. But we can apply Theorem 1 to determine where all the zeros of the polynomial lie. Using MATLAB, we obtain the following zeros : $-0.9154 + 0.4962i$, $-0.9154 - 0.4962i$, $0.0530 + 0.8845i$, $0.0530 - 0.8845i$, 0.3916 . Taking $\alpha = 2$ and $\beta = 0$, Theorem 1 gives that all the zeros of the polynomial lie in the closed disk $|3z + 2| \leq 7$.

Example 2. Next, consider

$$q(z) = -z^6 + 2z^5 + 2z^4 + 3z^3 + z^2 - 2.$$

The coefficients of $q(z)$ are $a_6 = -1$, $a_5 = 2$, $a_4 = 2$, $a_3 = 3$, $a_2 = 1$, $a_1 = 0$ and $a_0 = -2$. Using MATLAB, we obtain the following zeros: 3.0197 , $-0.7682 + 0.5814i$, $-0.7682 - 0.5814i$, $-0.0803 + 1.0233i$, $-0.0803 - 1.0233i$, 0.6773 . Taking $\lambda = 3$, $t = 1$ and $s = 0$, Theorem 3 gives that the zeros lie in $|z - 2| \leq 9$.

5 Acknowledgment

The author is greatly indebted to the referee for his/her several useful suggestions and valuable comments.

References

- [1] N. Anderson, E. B. Saff, and R. S. Varga, *On the Eneström-Kakeya theorem and its sharpness*, Linear Algebra and Its Applications **28** (1979), 5-16.
- [2] N. Anderson, E. B. Saff, and R. S. Varga, *An extension of the Eneström-Kakeya theorem and its sharpness*, SIAM J.Math. Anal. **12** (1981), 10-22.
- [3] A. Aziz and B. A. Zargar, *Bounds for the zeros of a polynomial with restricted coefficients*, Applied Math., (Irvine) **3** (2012), 30-33.

- [4] A. Aziz and B. A. Zargar, *Some extensions of Eneström-Kakeya Theorem*, Glas. Mat. Ser. III **31(51)** (1996), 239-244.
- [5] G. Eneström, *Härledning af en allmän formel för antalet pensionärer ...*, Öfv. af. Kungl. Vetenskaps-Akademiens Förhandlingar **6** (Stockholm, 1893).
- [6] A. Joyal, G. Labelle, and Q. I. Rahman, *On the location of Zeros of a Polynomial*, Canada Math. Bull. **10** (1967), 55-63.
- [7] S. Kakeya, *On the limits of the roots of an algebraic equation with positive coefficients*, Tôhoku Math. J. **2** (1912-13), 140-142.
- [8] M. Kovačević and I. Milovanović, *On a generalization of the Eneström-Kakeya Theorem*, Pure Math. and Appl., Ser. A **3** (1992), 43-47.